

Some properties of Graph Laplacians of cyclic groups

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Abstract

In this paper we investigate a spectra of the Laplacian matrix of cyclic groups using the properties of their characteristic polynomials. We have proved several assertions about the relationship between the spectra of different groups.

Keywords: Graph Laplacians, cyclic groups.

1 Introduction

Let us consider a graph G with the vertex set $V = \{1, \dots, n\}$ and the edge set E .

Definition 1.1. *The Laplacian matrix of the Graph G is a matrix $L(G) = (a_{i,j \in V})$, with*

$$a_{i,j} = \begin{cases} -1 & \text{if } ij \in E \\ d(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where $d(i) = |\{e \in E | i \in e\}|$ is the degree of the vertex i .

Definition 1.2. *The Cayley Graph of a discrete group L with a system of generators S is the graph whose vertices are the elements of the group L and whose edges are determined by the following condition: if g and s belong to L then there is an edge from g to f if and only if $f = g * s$ for some $s \in S \cup S^{-1}$.*

Let us consider the Cayley graph of the group Z_n . Note that the Laplacian is a nonnegative operator so all eigenvalues are greater or equal to 0. If $n = 1$, then the Laplacian of the Cayley graph of this group is $\begin{pmatrix} 0 \end{pmatrix}$. This matrix has only one eigenvalue which is zero. If $n = 2$, then the Laplacian of the Cayley graph of Z_2 is the matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

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The eigenvalues of the Laplacian are $\lambda = 0$ and $\lambda = 2$. The Laplacian of $Z_n, n > 3$ is the next matrix:

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}}_n$$

This matrix is circulant and its eigenvalues are known. In this paper we use another method instead of the well-known method of Gray (see [1]) by using spectrum to investigate the properties of the spectra and the characteristic polynomials of the Laplacians of cyclic groups. Let find the determinant of the following matrix. Set $a = 2 - \lambda$.

$$\begin{aligned} A_n &:= \underbrace{\begin{vmatrix} a & -1 & 0 & \dots & 0 & -1 \\ -1 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ -1 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_n = \underbrace{\begin{vmatrix} a & -1 & 0 & \dots & 0 & 0 \\ -1 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ 0 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_{n-1} + \\ &\underbrace{\begin{vmatrix} -1 & -1 & 0 & \dots & 0 & 0 \\ 0 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ -1 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_{n-1} + (-1)^n \underbrace{\begin{vmatrix} -1 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & a \\ -1 & 0 & 0 & \dots & 0 & -1 \end{vmatrix}}_{n-1} \\ L_{n-1} &:= \underbrace{\begin{vmatrix} a & -1 & 0 & \dots & 0 & 0 \\ -1 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ 0 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_{n-1} = a \underbrace{\begin{vmatrix} a & -1 & 0 & \dots & 0 & 0 \\ -1 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ 0 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_{n-2} + \end{aligned}$$

$$\begin{aligned}
& \underbrace{\begin{vmatrix} -1 & -1 & 0 & \dots & 0 & 0 \\ 0 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ 0 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_{n-1} = aL_{n-2} - L_{n-3} \quad (1) \\
& \underbrace{\begin{vmatrix} -1 & -1 & 0 & \dots & 0 & 0 \\ 0 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ -1 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_{n-1} = -L_{n-2} + \underbrace{\begin{vmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ -1 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_{n-2} = \\
& -L_{n-2} + \underbrace{\begin{vmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & -1 \\ -1 & 0 & 0 & \dots & -1 & a \end{vmatrix}}_{n-3} = \dots = -L_{n-2} + \begin{vmatrix} 0 & -1 \\ -1 & a \end{vmatrix} = -L_{n-2} - 1 \\
& \underbrace{\begin{vmatrix} -1 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & a \\ -1 & 0 & 0 & \dots & 0 & -1 \end{vmatrix}}_{n-1} = \underbrace{\begin{vmatrix} -1 & a & -1 & \dots & 0 & 0 \\ 0 & -1 & a & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & a \\ 0 & 0 & 0 & \dots & 0 & -1 \end{vmatrix}}_{n-2} + (-1)^n L_{n-2}
\end{aligned}$$

So

$$A_n = aL_{n-1} - 2L_{n-2} - 2 \quad (2)$$

Let complete the table of coefficients of L_n .

	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9
L_1	0	1	0	0	0	0	0	0	0	0
L_2	-1	0	1	0	0	0	0	0	0	0
L_3	0	-2	0	1	0	0	0	0	0	0
L_4	1	0	-3	0	1	0	0	0	0	0
L_5	0	3	0	-4	0	1	0	0	0	0
L_6	-1	0	6	0	-5	0	1	0	0	0
L_7	0	-4	0	10	0	-6	0	1	0	0
L_8	1	0	-10	0	15	0	-7	0	1	0
L_9	0	5	0	-20	0	21	0	-8	0	1

We can complete the table of coefficients of A_n from (2) and the table of coefficients of L_n .

	1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}
A_3	-2	-3	0	1	0	0	0	0	0	0	0	0
A_4	0	0	-4	0	1	0	0	0	0	0	0	0
A_5	-2	5	0	-5	0	1	0	0	0	0	0	0
A_6	-4	0	9	0	-6	0	1	0	0	0	0	0
A_7	-2	-7	0	14	0	-7	0	1	0	0	0	0
A_8	0	0	-16	0	20	0	-8	0	1	0	0	0
A_9	-2	9	0	-30	0	27	0	-9	0	1	0	0
A_{10}	-4	0	25	0	-50	0	35	0	-10	0	1	0
A_{11}	-2	-11	0	55	0	-77	0	44	0	-11	0	1

2 Main results

Lemma 2.1. $\forall n \in N, \forall k \subseteq [1, \dots, n] : L_n = L_{n-k}L_k - L_{n-k-1}L_{k-1}$.

Proof. $L_n = aL_{n-1} - L_{n-2} = a(aL_{n-2} - L_{n-3}) - L_{n-2} = (a^2 - 1)L_{n-2} - aL_{n-3} = L_2L_{n-2} - L_1L_{n-3}$.

Assume $L_n = L_kL_{n-k} - L_{k-1}L_{n-k-1}$. Then $L_n = L_kL_{n-k} - L_{k-1}L_{n-k-1} = L_k(aL_{n-k-1} - L_{n-k-2}) - L_{k-1}L_{n-k-1} = (aL_k - L_{k-1})L_{n-k-1} - L_kL_{n-k-2} = L_{k+1}L_{n-k-1} - L_kL_{n-k-2}$. \square

Lemma 2.2. $\forall n \in N : L_{n-1}^2 = L_{n-2}L_n + 1$.

Proof. $L_1 = a, L_2 = a^2 - 1, L_3 = a^3 - 2a, L_2^2 = L_3L_1 + 1$. Assume $L_{k-1}^2 = L_{k-2}L_k + 1$. Then $L_k^2 = L_{k-1}L_{k+1} + 1$; $L_{k-1}^2 = L_{k-2}L_k + 1$; $L_{k-1}^2 = L_{k-2}(aL_{k-1} - L_{k-2}) + 1$ and

$$L_{k-1}^2 + L_{k-2}^2 - 1 = aL_{k-1}L_{k-2} \quad (1)$$

$L_k^2 = L_{k-1}L_{k+1} + 1$; $(aL_{k-1} - L_{k-2})^2 = L_{k-1}(aL_k - L_{k-1}) + 1$; $a^2L_{k-1}^2 - 2aL_{k-1}L_{k-2} + L_{k-2}^2 = aL_kL_{k-1} - L_{k-1}^2 + 1$; $a^2L_{k-1}^2 - 2aL_{k-1}L_{k-2} = aL_kL_{k-1} - L_{k-1}^2 + 1$. Then by Equation (1) $aL_{k-1}^2 - aL_{k-1}L_{k-2} - aL_kL_{k-1} = 0$; $aL_{k-1}(aL_{k-1} - L_{k-2}) - aL_kL_{k-1} = 0$; $aL_kL_{k-1} - aL_kL_{k-1} = 0$. \square

Lemma 2.3. *If λ is the eigenvalue of Laplacian of Z_n , then λ is the eigenvalue of the Laplacian of $Z_{2^k n}$, $\forall k \in N$.*

Proof. $A_{2n} = aL_{2n-1} - 2L_{2n-2} - 2$. Then by Lemma 2.1 we have $L_{2n-1} = L_n L_{n-1} - L_{n-1} L_{n-2}$ and $A_{2n} = aL_n L_{n-1} - aL_{n-1} L_{n-2} - 2L_{2n-2} - 2 = a^2 L_{n-1}^2 - 2aL_{n-1} L_{n-2} - 2L_{2n-2} - 2$. The following are routine calculations: $L_{2n-2} = L_{n-1}^2 - L_{n-2}^2$; $A_{2n} = a^2 L_{n-1}^2 - 2aL_{n-1} L_{n-2} - 2L_{n-1}^2 + 2L_{n-2}^2 - 2$; $A_{2n} = (a^2 L_{n-1}^2 - 4aL_{n-1} L_{n-2} + 4L_{n-2}^2 - 4) - 2L_{n-1}^2 + 2aL_{n-1} L_{n-2} + 2$. Then by Lemma 2.2 $-2L_{n-1}^2 - 2L_{n-1}^2 + 2aL_{n-1} L_{n-2} + 2 = 0$. So $A_{2n} = a^2 L_{n-1}^2 - 4aL_{n-1} L_{n-2} + 4L_{n-2}^2 - 4 = (aL_{n-1} - 2L_{n-2})^2 - 4 = (aL_{n-1} - 2L_{n-2} - 2)(aL_{n-1} - 2L_{n-2} + 2) = A_n(A_n + 4)$.

Note that $A_2 = a^2 - 4$ is not the determinant of the Laplacian of Z_2 and $A_1 = a - 2$ is not the determinant of the Laplacian of the trivial group E . But $\lambda = 0$ is the eigenvalue of all Laplacian because each Laplacian is a singular matrix. However, by the Table 2 we see that $\lambda = 2$ is the eigenvalue of Laplacian with the multiplicity 2 of Z_{4k} , $k \in N$. □

Note that $\lambda = 2$ is not the eigenvalue of the Laplacian of Z_{4k-2} , $k \in N$. It is easy to see that $A_{4k-2}(0) = -2$.

Theorem 2.1. *If λ is the eigenvalue of the Laplacian of Z_n , $n \geq 3$, then λ is the eigenvalue of the Laplacian of Z_{kn} , $\forall k \in N$.*

Proof. Lemma 2.3 yields that A_n is a divisor of A_{2n} . Now suppose A_n is a divider of A_{mn} , $\forall m \leq k$.

$A_n = aL_{n-1} - 2L_{n-2} - 2 = L_n - L_{n-2} - 2$; $A_{(k+1)n} = L_{(k+1)n} - L_{(k+1)n-2} - 2 = L_{kn} L_n - L_{kn-1} L_{n-1} - L_{kn} L_{n-2} + L_{kn-1} L_{n-3} - 2 = L_{kn}(L_n - L_{n-2} - 2) + 2L_{kn} - 2 + L_{kn-1} L_{n-3} - L_{kn-1} L_{n-1} = A_n L_{kn} + 2L_{kn} - 2 + L_{kn-1} L_{n-3} - aL_{kn-2} L_{n-1} + L_{kn-3} L_{n-1} + L_{kn-2} L_{n-2} - L_{kn-2} L_{n-2} = A_n L_{kn} + 2L_{kn} - 2 + L_{kn-1} L_{n-3} + L_{kn-3} L_{n-1} - (L_{kn-2} L_n - L_{kn-2} L_{n-2} - 2L_{kn-2}) - 2L_{kn-2} - 2L_{kn-2} L_{n-2} = A_n L_{kn} - A_n L_{kn-2} + 2(L_{kn} - L_{kn-2} - 2) + 2 + L_{kn-1} L_{n-3} + L_{kn-3} L_{n-1} - 2L_{kn-2} L_{n-2} = A_n L_{kn} - A_n L_{kn-2} + 2A_{kn} + B$.

$B = 2 + L_{kn-1} L_{n-3} + L_{kn-3} L_{n-1} - 2L_{kn-2} L_{n-2} = 2 + L_{kn-2} L_{n-4} + L_{(k+1)n-4} + L_{kn-4} L_{n-2} + L_{(k+1)n-4} - 2L_{kn-3} L_{n-3} - 2L_{(k+1)n-4} = \dots = 2 + L_{(k-1)n+3} L_1 + L_{(k-1)n+1} L_3 - 2L_{(k-1)n+2} L_2 = 2 + L_{(k-1)n+2} + L_{(k-1)n+4} + L_{(k-1)n} L_2 + L_{(k-1)n+4} - 2L_{(k-1)n+1} L_1 - 2L_{(k-1)n+4} = (a^2 - 1)L_{(k-1)n} - aL_{(k-1)n-1} + (a^2 - 1)L_{(k-1)n} - 2a^2 L_{(k-1)n} + 2aL_{(k-1)n-1} + 2 = -2L_{(k-1)n} + aL_{(k-1)n-1} - L_{(k-1)n-2} + L_{(k-1)n-2} + 2 = -L_{(k-1)n} + L_{(k-1)n-2} + 2 = -A_{(k-1)n}$.

So $A_{(k+1)n} = A_n L_{kn} - A_n L_{kn-2} + A_{kn} - A_{(k-1)n} = (A_n + 2)A_{kn} + 2A_n - A_{(k-1)n}$ and A_n is a divider of $A_{(k+1)n}$. □

For example we can prove that the Laplacian spectra of $Z_2 \times Z_3$ and Z_6 are different. The graph of $Z_2 \times Z_3$ is isomorphic to the complement of the graph of Z_6 . It is well known that if $\lambda \neq 0$ is the eigenvalue of $L(G)$, then $n - \lambda$ is the eigenvalue of $L(G^C)$, see [2]. Since $\lambda = 4$ is the eigenvalue of Z_6 it follows that $\lambda = 2$ is the eigenvalue of $Z_2 \times Z_3$ and $\lambda = 2$ is not the eigenvalue of Z_6 . Therefore, the spectra of isomorphic groups can be different. Note that $Z_2 \times Z_2 \not\cong Z_4$ but their graphs and Laplacian spectra coincide.

Lemma 2.4. $A_{kn+p} = (A_p + 2)A_{kn} + 2A_p - A_{kn-p}$.

Proof. $A_{kn+p} = L_{kn+p} - L_{kn+p-2} - 2 = L_{kn} L_p - L_{kn-1} L_{p-1} - L_{kn} L_{p-2} + L_{kn-1} L_{p-3} - 2 = L_{kn}(L_p - L_{p-2} - 2) + 2L_{kn} - 2 + L_{kn-1} L_{p-3} - L_{kn-1} L_{p-1} = A_p L_{kn} + 2L_{kn} - 2 + L_{kn-1} L_{p-3} - aL_{kn-2} L_{p-1} + L_{kn-3} L_{p-1} +$

$$L_{kn-2}L_{p-2} - L_{kn-2}L_{p-2} = A_pL_{kn} + 2L_{kn} - 2 + L_{kn-1}L_{p-3} + L_{kn-3}L_{p-1} - (L_{kn-2}L_p - L_{kn-2}L_{p-2} - 2L_{kn-2}) - 2L_{kn-2} - 2L_{kn-2}L_{p-2} = A_pL_{kn} - A_pL_{kn-2} + 2(L_{kn} - L_{kn-2} - 2) + 2 + L_{kn-1}L_{p-3} + L_{kn-3}L_{p-1} - 2L_{kn-2}L_{p-2} = A_pL_{kn} - A_pL_{kn-2} + 2A_{kn} + B.$$

$$B = 2 + L_{kn-1}L_{p-3} + L_{kn-3}L_{p-1} - 2L_{kn-2}L_{p-2} = 2 + L_{kn-2}L_{p-4} + L_{(k+1)n-4} + L_{kn-4}L_{p-2} + L_{(k+1)n-4} - 2L_{kn-3}L_{p-3} - 2L_{(k+1)n-4} = \dots = 2 + L_{(k-1)n+3+(n-p)}L_1 + L_{(k-1)n+1+(n-p)}L_3 - 2L_{(k-1)n+2+(n-p)}L_2 = 2 + L_{(k-1)n+2+(n-p)} + L_{(k-1)n+4+(n-p)} + L_{(k-1)n+(n-p)}L_2 + L_{(k-1)n+4+(n-p)} - 2L_{(k-1)n+1+(n-p)}L_1 - 2L_{(k-1)n+4+(n-p)} = \dots = -A_{(k-1)n+(n-p)}.$$

$$\text{So } A_{kn+p} = A_pL_{kn} - A_pL_{kn-2} + 2A_{kn} - A_{(k-1)n+(n-p)} = (A_n + 2)A_{kn} + 2A_p - A_{kn-p}.$$

By Theorem 2.1 we get

$$A_{n+p} = A_n(A_p + 2) + 2A_p - A_{n-p}, p < n \quad (4)$$

□

Theorem 2.2. *If $\lambda \neq 2$ is the eigenvalue of Laplacian of Z_n and Z_m , then λ is the eigenvalue of the Laplacian of Z_d , where d is the greatest common divisor of m and n . Moreover, If $\lambda = 4$ is the eigenvalue of the Laplacian of Z_n , then $\exists k \in N : n = 2k$. Also, if $\lambda = 2$ is the eigenvalue of the Laplacian of Z_n , then $\exists k \in N : n = 4k$ or $n = 2$.*

Proof. Note that if $\lambda = 4$, then $A_2(2 - \lambda) = 0$. Assume that $\lambda \neq 2$ is the eigenvalue of the Laplacian of Z_n and Z_m when $m > n, m = n + k$, and that the greatest common divisor of m and n is 1. Set $a = 2 - \lambda$. Then $A_{n+k}(a) = A_n(a) = 0$. By the (4) we have: $A_{2n+k}(a) = A_{n+k}(a)(A_n(a) + 2) + 2A_n(a) - A_k(a) = -A_k(a)$ $A_{2n+2k}(a) = A_{2n+k}(a)(A_k(a) + 2) + 2A_{2n}(a) - A_k(a) = -A_k^2(a)$. But $A_{2n+2k}(a) = A_{2(n+k)}(a) = 0$ by the (4.1). So $A_k(a) = 0$. If k and n have the common divisor > 1 then m and n have the common divisor > 1 too. So the greatest common divisor of k and $\min(n, n+k)$ is 1. Continuing this procedure for the k and $\min(n, n+k)$ we obtain the following:

$$A_{\min(k, \min(n, n+k))}(a) = A_{|k - \min(n, n+k)|}(a) = 0$$

In addition, the greatest common divisor of $\min(k, \min(n, n+k))$ and $|k - \min(n, n+k)|$ is 1. Continuing this procedure further we prove for some p that $A_p(a) = A_1(a) = 0$. So if the greatest common divisor of m and n is 1, then $A_m(a) = A_n(a) = A_1(a) = 0 \Rightarrow a = 2$ and $\lambda = 0$. □

Note then the multiplicity of the first eigenvalue $\lambda = 0$ is equal to the number of components of graph (see [3],[2]). So for all cyclic groups the multiplicity of $\lambda = 0$ is 1.

Lemma 2.5. $A_n(a) = aA_{n-1}(a) - A_{n-2}(a) + 2A_1(a), n \geq 3$

$$\text{Proof. } A_n = aL_{n-1} - 2L_{n-2} - 2 = a(aL_{n-2} - L_{n-3}) - 2L_{n-2} - 2 = a^2L_{n-2} - L_{n-3} - 2L_{n-2} - 2 = a^2L_{n-2} - 2aL_{n-3} - 2a + aL_{n-3} + 2a - 2L_{n-2} - 2 = a(aL_{n-2} - 2L_{n-3} - 2) + aL_{n-3} + 2a - 2L_{n-2} - 2 = aA_{n-1} + aL_{n-3} + 2a - 2 - 2(aL_{n-3} - L_{n-4}) = aA_{n-1} - aL_{n-3} + 2L_{n-4} + 2 + 2a - 4 = aA_{n-1} - A_{n-2} + 2A_1. \quad \square$$

Lemma 2.6. $A_{kn} = A_k \circ (A_n + 2)$.

Proof. $A_{2n} = A_n(A_n + 4) = (A_n + 2 - 2)(A_n + 4) = (A_n + 2)^2 - 4 = A_2 \circ (A_n + 2)$. Now assume $\forall m \leq k : A_{mn} = A_m \circ (A_n + 2)$.

$$A_{(k+1)n}(a) = A_{kn}(a)(A_n(a) + 2) + 2A_n(a) - A_{(k-1)n}(a) = (A_n(a) + 2)A_k \circ (A_n(a) + 2) - A_{(k-1)n} \circ (A_n(a) + 2)$$

$$2) + 2(A_n(a) + 2) - 4 = (A_n(a) + 2)A_k \circ (A_n(a) + 2) - A_{(k-1)n} \circ (A_n(a) + 2) + 2A_1 \circ (A_n(a) + 2).$$

Hence by Lemma 2.5, $A_{(k+1)n}(a) = A_{k+1} \circ (A_n(a) + 2)$. \square

Theorem 2.3. *If λ is the eigenvalue of the Laplacian of Z_n , $n \geq 3$ with the multiplicity r , then λ is the eigenvalue of the Laplacian of Z_{kn} , $\forall k \in N$ with the multiplicity r . Furthermore, If $\lambda = 4$ is eigenvalue of the Laplacian of Z_n , then $\exists k \in N : n = 2k$ and the multiplicity of λ is 1. Also if $\lambda = 2$ is the eigenvalue of the Laplacian of Z_n , $n > 2$, then $\exists k \in N : n = 4k$ and the multiplicity of $\lambda = 2$ is 2.*

Proof. Assume that λ_0 is the eigenvalue of the Laplacian of Z_n with the multiplicity r and that of the Laplacian of Z_{kn} with the multiplicity q . Put $a_0 = 2 - \lambda_0$. Obviously $r \leq q$. Now suppose $r < q$. Then by Lemma 2.6 we have $A_{kn} = A_k \circ (A_n + 2) = \prod_{i=1}^k (A_n + 2 - a_i)$, where a_i are the roots of $A_k(a) = 0$. Note that $\exists! a_i : A_n + 2 - a_i = 0$ and $a_i = 2$. Since $r \leq q$ then $A_{kn}/A_n = \prod_{j=1}^{k-1} (A_n(a) + 2 - a_j) = 0$, where a_j are the roots of $A_k(a) = 0$ and $\forall j : a_j \neq 2$. $A_n(a) = 0 \Rightarrow A_{kn}/A_n = \prod_{j=1}^{k-1} (2 - a_j) = 0$. But $\forall j : a_j \neq 2$. So $r \not\leq q \Rightarrow r = q$. \square

Theorem 2.4. *If $\lambda_0 \neq 2$ is the eigenvalue of the Laplacian of Z_n , then $\forall m \in N : P_m(\lambda_0) = -A_m(2 - \lambda_0) = \lambda_1$, where λ_1 is the eigenvalue of the Laplacian of Z_n .*

Proof. By Lemma 2.6 we get $A_{mn}(2 - \lambda_0) = \prod_{j=1}^n (A_m(2 - \lambda_0) + 2 - (2 - \lambda_j)) = \prod_{j=1}^n (A_m(2 - \lambda_0) + \lambda_j) = 0$. Thus, $\exists \lambda_1 : P_m(\lambda_0) = -A_m(2 - \lambda_0) = \lambda_1$, where λ_1 is the eigenvalue of the Laplacian of Z_n . \square

Corollary 2.1. *If λ is the eigenvalue of Laplacian of Z_n , then $\lambda \in [0, 4]$.*

Proof. Since all $\lambda \geq 0$, then by Theorem 2.4 we have $\forall \lambda_0 : P_2(\lambda_0) = \lambda_0(4 - \lambda_0) \geq 0$. \square

Corollary 2.2. $P_k(\lambda) = \lambda_i$, where λ_i is the eigenvalue of the Laplacian of $Z_n \Leftrightarrow \lambda$ is the eigenvalue of the Laplacian of Z_{kn} .

Proof. By Lemma 2.6 we see that $P_{kn} = (-1)^{n-1}(P_k - \lambda_j)$, where λ_j are the eigenvalues of the Laplacian of Z_n . \square

References

- [1] Gray R.M. Toeplitz and Circulant Matrices: A review. Now Publishers Inc, 2006.
- [2] Turker Biyikoglu, Josef Leydold, Peter F. Stadler. Laplacian Eigenvectors of Graphs: Frobenius and Faber-Krahn Type Theorems. Springer, 2007.
- [3] D. Cvetkovic, M. Doob, I. Gutman, and A. Torgasev. Recent results in the theory of graph spectra, Ann. Discr. Math. 36, North Holland, 1988.